Published by Institute of Physics Publishing for SISSA
Received: *May 24, 2005*
Accepted: *June 2, 2005*
Published: *June 30, 2005*

Dualities and signatures of $\mathcal{G}^{++}$-invariant theories

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Abstract: The $\mathcal{G}^{++}$-content of the formulation of gravity and M-theories as very-extended Kac-Moody invariant theories is further analysed. The different exotic phases of all the $\mathcal{G}_B^{++}$-theories, which admit exact solutions describing intersecting branes smeared in all directions but one, are derived. This is achieved by analysing for all $\mathcal{G}^{++}$ the signatures which are related to the conventional one $(1, D-1)$ by 'dualities' generated by the Weyl reflections.

Keywords: Space-Time Symmetries, p-branes, String Duality, Classical Theories of Gravity.

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JHEP06(2005)084

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1. Introduction and discussion

A theory containing gravity suitably coupled to forms and dilatons may exhibit upon dimensional reduction down to three dimensions a simple Lie group $\mathcal{G}$ symmetry non-linearly realised. The scalars of the dimensionally reduced theory live in a coset $\mathcal{G}/\mathcal{H}$ where $\mathcal{G}$ is in its maximally non-compact form and $\mathcal{H}$ is the maximal compact subgroup of $\mathcal{G}$. A maximally oxidised theory is such a lagrangian theory defined in the highest possible space-time dimension D namely a theory which is itself not obtained by dimensional reduction.

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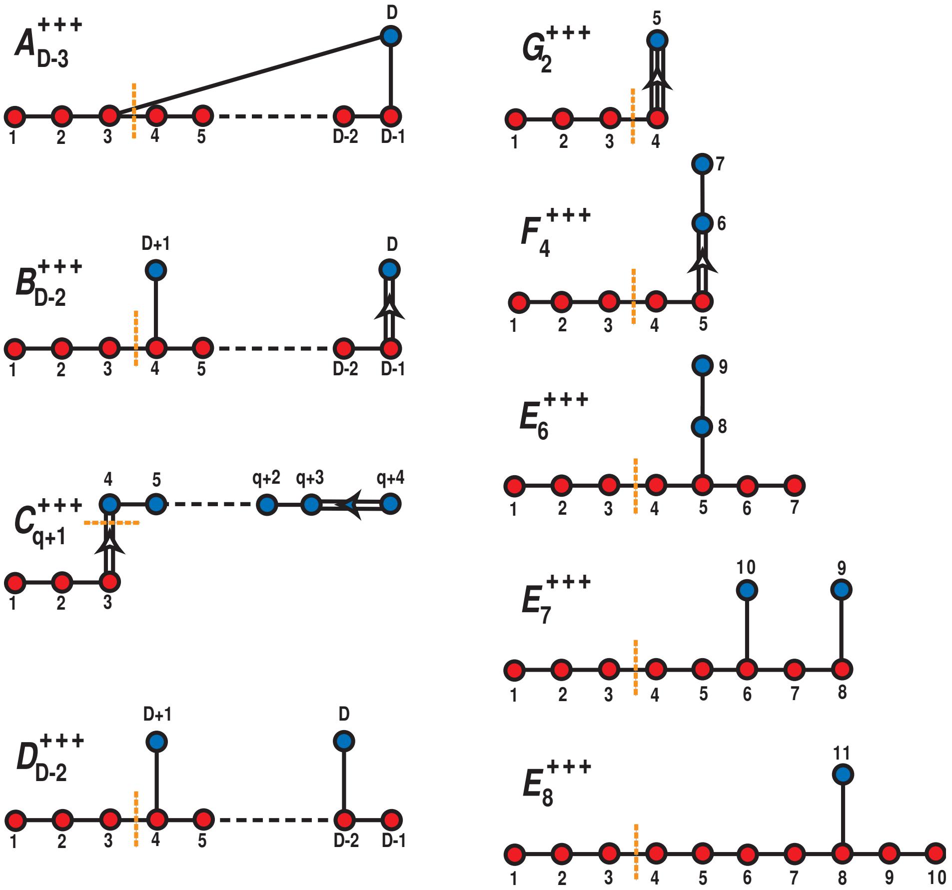

Figure 1: The nodes labelled 1,2,3 define the Kac-Moody extensions of the Lie algebras. The horizontal line starting at 1 defines the 'gravity line', which is the Dynkin diagram of a A_{D-1} subalgebra.

These maximally oxidised actions have been constructed for all $\mathcal{G}$ [1] and they include in particular pure gravity in D dimensions and the low energy effective actions of the bosonic string and of M-theory.

It has been conjectured that these theories, or some extensions of them, possess the much larger very-extended Kac-Moody symmetry $\mathcal{G}^{+++}$. $\mathcal{G}^{+++}$ algebras are defined by the Dynkin diagrams depicted in figure 1, obtained from those of $\mathcal{G}$ by adding three nodes [2]. One first adds the affine node, labelled 3 in the figure, then a second node, 2, connected to it by a single line and defining the overextended $\mathcal{G}^{++}$ algebra,[1] then a third one, 1, connected by a single line to the overextended node. Such $\mathcal{G}^{+++}$ symmetries were first conjectured in the aforementioned particular cases [4, 5] and the extension to all $\mathcal{G}^{+++}$ was proposed in [6]. In a different development, the study of the properties of cosmological solutions in the vicinity of a space-like singularity, known as cosmological billiards [7], revealed an overextended symmetry $\mathcal{G}^{++}$ for all maximally oxidised theories [8, 9].

[1] In the context of dimensional reduction, the appearance of $E_8^{++} = E_{10}$ in one dimension has been first conjectured by B. Julia [3].

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The possible existence of this Kac-Moody symmetry $\mathcal{G}^{+++}$ motivated the construction of a lagrangian formulation explicitly invariant under $\mathcal{G}^{+++}$ [10]. The action $S_{\mathcal{G}^{+++}}$ is defined in a reparametrisation invariant way on a world-line, a priori unrelated to space-time, in terms of fields $\phi(\xi)$ living in a coset $\mathcal{G}^{+++}/K^{+++}$ where ξ spans the world-line. A level decomposition of $\mathcal{G}^{+++}$ with respect to the subalgebra A_{D-1} of its gravity line (see figure 1) is performed where D is identified to the space-time dimension.[2] The subalgebra K^{+++} is invariant under a 'temporal' involution which ensures that the action is $SO(1, D-1)$ invariant at each level where the index 1 of A_{D-1} is identified to a time coordinate.

The $\mathcal{G}^{++}$ content of the $\mathcal{G}^{+++}$ -invariant actions $S_{\mathcal{G}^{+++}}$ has been analysed in reference [14] where it was shown that two distinct actions invariant under the overextended Kac-Moody algebra $\mathcal{G}^{++}$ exist. The first one $S_{\mathcal{G}_C^{++}}$ is constructed from $S_{\mathcal{G}^{+++}}$ by performing a truncation putting consistently to zero some fields. The corresponding $\mathcal{G}^{++}$ algebra is obtained from $\mathcal{G}^{+++}$ by deleting the node labelled 1 from the Dynkin diagram of $\mathcal{G}^{+++}$ depicted in figure 1. This theory carries a euclidean signature and is the generalisation to all $\mathcal{G}^{++}$ of the $E_8^{++} = E_{10}$ invariant action of reference [15] proposed in the context of M-theory and cosmological billiards. The parameter ξ is then identified with the time coordinate and the action restricted to a defined number of lowest levels is equal to the corresponding maximally oxidised theory in which the fields depend only on this time coordinate. A second $\mathcal{G}^{++}$-invariant action $S_{\mathcal{G}_B^{++}}$ is obtained from $S_{\mathcal{G}^{+++}}$ by performing the same consistent truncation *after* conjugation by the Weyl reflection W_{α_1} in $\mathcal{G}^{+++}$ where W_{α_1} is the Weyl reflection in the hyperplane perpendicular to the simple root α_1 corresponding to the node 1 of figure 1. The non-commutativity of the temporal involution with the Weyl reflection [16, 17] implies that this second action is inequivalent to the first one. In $S_{\mathcal{G}_B^{++}}$, ξ is identified with a space-like direction and the theory admits exact solutions which are identical to those of the corresponding maximally oxidised theory describing intersecting extremal brane configurations smeared in all directions but one [14, 10, 18]. Furthermore the intersection rules [19] are neatly encoded in the $\mathcal{G}^{++}$ algebra through orthogonality conditions between the real positive roots corresponding to the branes in the configuration [18].

The Weyl reflections of $\mathcal{G}^{+++}$ generated by roots not belonging to the gravity line are also Weyl reflection of $\mathcal{G}^{++}$, their actions on $S_{\mathcal{G}_B^{++}}$ are thus well defined. These reflections yield actions with different global signatures which are related by field redefinitions [14]. This equivalence realises in the action formalism the general analysis of Keurentjes [16, 17].

The precise analysis of the different possible signatures has been performed for $\mathcal{G}_B^{++} = E_8^{++} = E_{10}$. In this case the corresponding maximally oxidised theory is the bosonic sector of the low effective action of M-theory. For E_8^{++} the Weyl reflection $W_{\alpha_{11}}$ generated by the simple root α_{11} (see figure 1) corresponds to a double T-duality in the directions 9 and 10 followed by an exchange of the two radii [20, 21, 6]. The signatures found in the analysis of references [16, 17] and in the context of $S_{\mathcal{G}_B^{++}}$ in [14] match perfectly with the signature changing dualities and the exotic phases of M-theories discussed in [22, 23].

[2]Level expansions of very-extended algebras in terms of the subalgebra A_{D-1} have been considered in [11 – 13].

The action of Weyl reflections generated by simple roots not belonging to the gravity line on the exact extremal brane solutions has been studied for all $\mathcal{G}^{+++}$ -theory constructed with the temporal involution selecting the index 1 as a time coordinate [10]. The existence of Weyl orbits of extremal brane solutions similar to the U-duality orbits existing in M-theory strongly suggests a general group-theoretical origin of 'dualities' for all $\mathcal{G}^{+++}$ -theories transcending string theories and supersymmetry.

In this context it is certainly interesting to extend to all $\mathcal{G}^{+++}$-theories the analysis of signature changing Weyl reflections. This is the purpose of the present work. We find for all the $\mathcal{G}_B^{++}$-theories all the possible signature (T, S), where T (resp. S) is the number of time-like (resp. space-like) directions, related by Weyl reflections of $\mathcal{G}^{++}$ to the signature $(1, D-1)$ associated to the theory corresponding to the traditional maximally oxidised theories. Along with the different signatures the signs of the kinetic terms of the relevant fields are also discussed. We start the analysis with A_{D-3}^{++} corresponding to pure gravity in D dimensions then we extend the analysis to the other $\mathcal{G}^{++}$, first to the simply laced ones and then to the non-simply laced ones.[3] Each $\mathcal{G}^{++}$ algebra contains a A_{D-3}^{++} subalgebra, the signatures of $\mathcal{G}^{++}$ should thus includes the one of A_{D-3}^{++}. This is indeed the case, but some $\mathcal{G}^{++}$ will contain additional signatures. If one want to restrict our focus on string theory, the special cases of D_{24}^{++} and B_8^{++} are interesting, the former being related to the low-energy effective action of the bosonic string (without tachyon) and the latter being related to the low-energy effective action of the heterotic string (restricted to one gauge field). The existence of signature changing dualities are related to the magnetic roots and suggests that these transformations correspond to a generalisation of the S-duality existing in these two theories [24].

The approach based on Kac-Moody algebras constitutes certainly a very-exciting and innovative attempt to understand gravitational theories encompassing string theories. It is certainly worthwhile to pursue this route by analysing further the structure of these $\mathcal{G}^{++} \subset \mathcal{G}^{+++}$ -theories and to try to understand if it could lead to a completely new formulation of gravitational interactions where the structure of space-time is hidden somewhere in these huge algebras [15, 14, 25] or even huger ones [26].

The paper is organised as follows. In section 2, we recall the construction of $S_{\mathcal{G}^{+++}}$-invariant actions. In section 3, we review the non-commutativity of the temporal involution with the Weyl reflections and develop the notation and tools necessary to discuss signature changes in our setting. In section 4, we review briefly the two different $\mathcal{G}^{++}$-theories obtained by truncation of the $S_{\mathcal{G}^{+++}}$ action. Finally in section 5, we derive all the possible signatures for all the $\mathcal{G}_B^{++}$-theories.

2. The $\mathcal{G}^{+++}$-invariant theories

In this section, we recall the construction of the $\mathcal{G}^{+++}$-invariant theories [10]. Actions $S_{\mathcal{G}^{+++}}$ invariant under non-linear transformations of $\mathcal{G}^{+++}$ are constructed recursively from a level decomposition with respect to a subalgebra A_{D-1} where D is interpreted as the

[3] Arjan Keurentjes informed us that he obtained independently similar results to be released soon.

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space-time dimension. Each $\mathcal{G}^{+++}$ contains indeed a subalgebra GL(D) such that $SL(D)(= A_{D-1}) \subset \mathrm{GL}(D) \subset \mathcal{G}^{+++}$. The action is defined in a reparametrisation invariant way on a world-line, a priori unrelated to space-time, in terms of fields $\varphi(\xi)$ where ξ spans the world-line. The fields $\varphi(\xi)$ live in a coset space $\mathcal{G}^{+++}/K^{+++}$ where the subalgebra K^{+++} is invariant under a 'temporal involution' preserving at each level a Lorentz algebra $SO(1, D-1) = A_{D-1} \cap K^{+++}$.

The generators of the GL(D) subalgebra are taken to be $K^a{}_b$ $(a, b = 1, 2, \ldots, D)$ with commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b \,. \tag{2.1}$$

The $K^a{}_b$ along with abelian generators R_u $(u = 1 \ldots q)$, which are present when the corresponding maximally oxidised action $S_{\mathcal{G}}$ has q dilatons,[4] are the level zero generators. The step operators of level greater than zero are tensors $R_{d_1 \ldots d_s}{}^{c_1 \ldots c_r}$ of the A_{D-1} subalgebra. Each tensor forms an irreducible representation of A_{D-1} characterised by some Dynkin labels. In principle it is possible to determine the irreducible representations present at each level [12, 13]. The lowest levels contain antisymmetric tensor step operators $R^{a_1 a_2 \ldots a_r}$ associated to electric and magnetic roots arising from the dimensional reduction of field strength forms in $S_{\mathcal{G}}$. They satisfy the tensor and scaling relations

$$[K^a{}_b, R^{a_1 \ldots a_r}] = \delta^{a_1}_b R^{a a_2 \ldots a_r} + \cdots + \delta^{a_r}_b R^{a_1 \ldots a_{r-1} a} \,, \tag{2.2}$$

$$[R, R^{a_1 \ldots a_r}] = -\frac{\varepsilon_A a_A}{2} R^{a_1 \ldots a_r} \,, \tag{2.3}$$

where a_A is the dilaton coupling constant to the field strength form and ε_A is $+1\,(-1)$ for an electric (magnetic) root [6]. The generators obey the invariant scalar product relations

$$\langle K^a{}_a K^b{}_b \rangle = G_{ab} \,, \quad \langle K^b{}_a K^d{}_c \rangle = \delta^b_c \delta^d_a \; a \neq b \,, \qquad \langle RR \rangle = \frac{1}{2} \,, \tag{2.4}$$

$$\langle R_{b_1 \ldots b_s}{}^{a_1 \ldots a_r}, \bar{R}_{d_1 \ldots d_r}{}^{c_1 \ldots c_s} \rangle = \delta^{c_1}_{b_1} \ldots \delta^{c_s}_{b_s} \delta^{a_1}_{d_1} \ldots \delta^{a_r}_{d_r} \,. \tag{2.5}$$

Here $G = I_D - \frac{1}{2}\Xi_D$ where Ξ_D is a D-dimensional matrix with all entries equal to unity and $\bar{R}_{d_1 \ldots d_r}{}^{c_1 \ldots c_s}$ designates the negative step operator conjugate to $R_{c_1 \ldots c_s}{}^{d_1 \ldots d_r}$.

The temporal involution Ω_1 generalises the Chevalley involution to allow identification of the index 1 to a time coordinate in $SO(1, D-1)$. It is defined by

$$K^a{}_b \overset{\Omega_1}{\mapsto} -\epsilon_a \epsilon_b K^b{}_a \quad R \overset{\Omega_1}{\mapsto} -R \quad , \quad R_{d_1 \ldots d_s}{}^{c_1 \ldots c_r} \overset{\Omega_1}{\mapsto} -\epsilon_{c_1} \ldots \epsilon_{c_r} \epsilon_{d_1} \ldots \epsilon_{d_s} \bar{R}_{c_1 \ldots c_r}{}^{d_1 \ldots d_s} \,, \tag{2.6}$$

with $\epsilon_a = -1$ if $a = 1$ and $\epsilon_a = +1$ otherwise. It leaves invariant a subalgebra K^{+++} of $\mathcal{G}^{+++}$.

The fields $\varphi(\xi)$ living in the coset space $\mathcal{G}^{+++}/K^{+++}$ parametrise the Borel group built out of Cartan and positive step operators in $\mathcal{G}^{+++}$. Its elements $\mathcal{V}$ are written as

$$\mathcal{V}(\xi) = \exp\left(\sum_{a \geq b} h_b{}^a(\xi) K^b{}_a - \sum_{u=1}^{q} \phi^u(\xi) R_u\right) \exp\left(\sum \frac{1}{r!s!} A_{b_1 \ldots b_s}{}^{a_1 \ldots a_r}(\xi) R_{a_1 \ldots a_r}{}^{b_1 \ldots b_s} + \cdots\right), \tag{2.7}$$

[4]All the maximally oxidised theories have at most one dilaton except the C_{q+1}-series characterised by q dilatons.

where the first exponential contains only level zero operators and the second one the positive step operators of levels strictly greater than zero. Defining

$$dv(\xi) = d\mathcal{V}\mathcal{V}^{-1} \qquad d\tilde{v}(\xi) = -\Omega_1\, dv(\xi) \qquad dv_{\text{sym}} = \frac{1}{2}(dv + d\tilde{v})\,, \tag{2.8}$$

one obtains, in terms of the ξ-dependent fields, an action $S_{\mathcal{G}^{+++}}$ invariant under global $\mathcal{G}^{+++}$ transformations, defined on the coset $\mathcal{G}^{+++}/K^{+++}$

$$S_{\mathcal{G}^{+++}} = \int d\xi \frac{1}{n(\xi)} \left\langle \left(\frac{dv_{\text{sym}}(\xi)}{d\xi} \right)^2 \right\rangle, \tag{2.9}$$

where $n(\xi)$ is an arbitrary lapse function ensuring reparametrisation invariance on the world-line.

Writing

$$S_{\mathcal{G}^{+++}} = S^{(0)}_{\mathcal{G}^{+++}} + \sum_A S^{(A)}_{\mathcal{G}^{+++}}\,, \tag{2.10}$$

where $S^{(0)}_{\mathcal{G}^{+++}}$ contains all level zero contributions, one obtains

$$S^{(0)}_{\mathcal{G}^{+++}} = \frac{1}{2}\int d\xi \frac{1}{n(\xi)} \left[\frac{1}{2}\left(g^{\mu\nu} g^{\sigma\tau} - \frac{1}{2} g^{\mu\sigma} g^{\nu\tau} \right) \frac{dg_{\mu\sigma}}{d\xi}\frac{dg_{\nu\tau}}{d\xi} + \sum_{u=1}^{q} \frac{d\phi^u}{d\xi}\frac{d\phi^u}{d\xi} \right], \tag{2.11}$$

$$S^{(A)}_{\mathcal{G}^{+++}} = \frac{1}{2r!s!}\int d\xi \frac{e^{-2\lambda\phi}}{n(\xi)} \left[\frac{DA_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}}{d\xi} g^{\mu_1\mu'_1}\ldots g^{\mu_r\mu'_r} g_{\nu_1\nu'_1}\ldots g_{\nu_s\nu'_s} \frac{DA_{\mu'_1\ldots\mu'_r}{}^{\nu'_1\ldots\nu'_s}}{d\xi} \right]. \tag{2.12}$$

The ξ-dependent fields $g_{\mu\nu}$ are defined as $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ where $e_\mu{}^a = (e^{-h(\xi)})_\mu{}^a$. The appearance of the Lorentz metric η_{ab} with $\eta_{11} = -1$ is a consequence of the temporal involution Ω_1. The metric $g_{\mu\nu}$ allows a switch from the Lorentz indices (a,b) of the fields appearing in eq. (2.7) to GL(D) indices (μ,ν). $D/D\xi$ is a covariant derivative generalising $d/d\xi$ through non-linear terms arising from non-vanishing commutators between positive step operators and λ is the generalisation of the scale parameter $-\varepsilon_A a_A/2$ to all roots.

3. Weyl reflections and the temporal involution

The non-commutativity of the temporal involution with the Weyl reflections implies that different space-time signatures are related between themselves [16, 17]. In this section, we recall the basic facts, we set up the notations and recollect some tools and formulae necessary for the general discussion of signature changes presented in section 5 in the setting of our construction of $\mathcal{G}^{++}$-actions [14].

A Weyl transformation W can be expressed as a conjugation by a group element U_W of $\mathcal{G}^{+++}$. We define the involution Ω' operating on the conjugate elements by

$$U_W\, \Omega T\, U_W^{-1} = \Omega'\, U_W T U_W^{-1}\,, \tag{3.1}$$

where T is any generator of $\mathcal{G}^{+++}$.

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gravity line	$K^1{}_2$	$K^2{}_3$	$K^3{}_4$	$\dots$	$K^{D-1}{}_D$	time coordinate
Ω	$+$	$-$	$-$	$-$	$-$	1
Ω'	$+$	$+$	$-$	$-$	$-$	2

Table 1: Involution switch from Ω to Ω' due to the Weyl reflection W_{α_1}.

We first recall the effect of the Weyl reflection W_{α_1} generated by the simple root α_1 (see figure 1). One gets

$$\begin{aligned} U_1\,\Omega K^2{}_1\,U_1^{-1} &= \rho K^2{}_1 = \rho\Omega' K^1{}_2\,, \\ U_1\,\Omega K^1{}_3\,U_1^{-1} &= \sigma K^3{}_2 = \sigma\Omega' K^2{}_3\,, \\ U_1\,\Omega K^i{}_{i+1}\,U_1^{-1} &= -\tau K^{i+1}{}_i = \tau\Omega' K^i{}_{i+1} \quad i>2\,. \end{aligned} \tag{3.2}$$

Here ρ, σ, τ are plus or minus signs which may arise as step operators are representations of the Weyl group up to signs. Eq. (3.2) illustrate the general result that such signs always cancel in the determination of Ω' . The content of eq. (3.2) is represented in table 1. The signs below the generators of the gravity line indicate the sign in front of the negative step operator obtained by the involution: a minus sign is in agreement with the conventional Chevalley involution and indicates that the indices in $K^a{}_{a+1}$ are both either space or time indices while a plus sign indicates that one index must be time and the other space. If we choose the description which leaves unaffected coordinates attached to planes invariant under the Weyl transformation, table 1 shows that the time coordinate must be identified with 2. The generic Weyl reflection W_{α_a} generated by α_a a simple root of the gravity line exchanges the index a and $a+1$ along with the space-time nature of the corresponding coordinates.

Weyl reflections generated by simple roots not belonging to the gravity line relate step operators of different levels. As a consequence [16, 17], these may potentially induce changes of signature far less trivial than the simple exchange of the index identifying the time coordinate . These changes have been studied from the algebraic point of view in great details for E_{11} (and more generally for E_n) in [16, 17].[5]

In order to address the action of an involution on a generic step operator $R^{a_1\dots a_r}$ of level greater than zero in a given irrreducible representation of A_{D-1}, we introduce some notations. First, given an involution $\tilde\Omega$, one defines $\mathrm{sign}(\tilde\Omega X)$ for any given positive step operator X in the following way

$$\tilde\Omega\, X \equiv \mathrm{sign}(\tilde\Omega X)\,\bar X\,, \tag{3.3}$$

where $\bar X$ designates the negative step operator conjugate to X. Second, we also introduce a sign associated to a given positive step operator of level greater than zero[6] $R^{a_1\dots a_r}$ in the following way

$$+ \quad : \mathrm{sign}(\tilde\Omega R^{a_1\dots a_r}) = -\epsilon_{a_1}\dots\epsilon_{a_r} \tag{3.4}$$

$$- \quad : \mathrm{sign}(\tilde\Omega R^{a_1\dots a_r}) = +\epsilon_{a_1}\dots\epsilon_{a_r}\,, \tag{3.5}$$

[5] Some algebraic considerations in this context for others groups $\mathcal{G}$ are presented in the Appendix of the second reference.

[6] As far as the action of the involution is concerned the symmetry properties of a step operator given by its Dynkin labels do not play any role.

where $\epsilon_a = -1$ if a is a timelike index and $\epsilon_a = +1$ if a is a space-like index, the space-time nature of the coordinate labelled by the index a being defined by the action of $\tilde{\Omega}$ on the $K^a_{\ b}$. The + sign defined in eq. (3.4) will lead to the positive kinetic energy term for the corresponding field in the action while the − sign defined in eq. (3.5) will lead to a negative kinetic energy term.

Finally, if we perform a Weyl reflection W_Y generated by a simple root not belonging to the gravity line and associated to a step operator Y eq. (3.1) gives

$$\mathrm{sign}(\Omega Y) = \mathrm{sign}(\Omega' Y)\,, \tag{3.6}$$

because $\mathrm{sign}(\tilde{\Omega} Y) = \mathrm{sign}(\tilde{\Omega}\bar{Y})$ where $\bar{Y}$ is the negative step operator conjugate to Y.

4. From the $\mathcal{G}^{+++}$-theory to the $\mathcal{G}_C^{++}$ and $\mathcal{G}_B^{++}$-theories

It has been recently shown [14] that for each very-extended algebra $\mathcal{G}^{+++}$, the $\mathcal{G}^{+++}$-invariant theory encompasses two distinct theories invariant under the overextended Kac-Moody subalgebra $\mathcal{G}^{++}$. The $\mathcal{G}_C^{++}$- invariant action $S_{\mathcal{G}_C^{++}}$ describes a motion in a coset $\mathcal{G}^{++}/K^{++}_{(+)}$ and carries a euclidean signature while the second theory described by a different embedding of $\mathcal{G}^{++}$ in $\mathcal{G}^{+++}$, referred as $\mathcal{G}_B^{++}$, describes a motion in a different coset $\mathcal{G}^{++}/K^{++}_{(-)}$. In contradistinction with the $\mathcal{G}_C^{++}$ case, the $\mathcal{G}_B^{++}$-theory carries various lorentzian signatures which are revealed through various equivalent formulation related by Weyl transformations. We now recall the construction of these two theories.

4.1 The $\mathcal{G}_C^{++}$-theory

Consider the overextended algebra $\mathcal{G}_C^{++}$ obtained from the very-extended algebra $\mathcal{G}^{+++}$ by deleting the node labelled 1 from the Dynkin diagrams of $\mathcal{G}^{+++}$ depicted in figure 1. The action $S_{\mathcal{G}_C^{++}}$ describing the $\mathcal{G}_C^{++}$ theory is obtained from $\mathcal{G}^{+++}$ by performing the following consistent truncation. One puts to zero in the coset representative eq. (2.7) the field multiplying the Chevalley generator $H_1 = K^1_{\ 1} - K^2_{\ 2}$ and all the fields multiplying the positive step operators associated to roots whose decomposition in terms of simple roots contains the deleted root α_1. Performing this truncation one obtains the following action

$$S_{\mathcal{G}_C^{++}} = S^{(0)}_{\mathcal{G}_C^{++}} + \sum_B S^{(B)}_{\mathcal{G}_C^{++}}\,, \tag{4.1}$$

where

$$S^{(0)}_{\mathcal{G}_C^{++}} = \frac{1}{2}\int dt \frac{1}{n(t)}\left[\frac{1}{2}(g^{\hat\mu\hat\nu}g^{\hat\sigma\hat\tau} - g^{\hat\mu\hat\sigma}g^{\hat\nu\hat\tau})\frac{dg_{\hat\mu\hat\sigma}}{dt}\frac{dg_{\hat\nu\hat\tau}}{dt} + \frac{d\phi}{dt}\frac{d\phi}{dt}\right], \tag{4.2}$$

$$S^{(B)}_{\mathcal{G}_C^{++}} = \frac{1}{2r!s!}\int dt \frac{e^{-2\lambda\phi}}{n(t)}\left[\frac{DB_{\hat\mu_1\ldots\hat\mu_r}{}^{\hat\nu_1\ldots\hat\nu_s}}{dt} g^{\hat\mu_1\hat\mu'_1}\ldots g^{\hat\mu_r\hat\mu'_r} g_{\hat\nu_1\hat\nu'_1}\ldots, g_{\hat\nu_s\hat\nu'_s}\frac{DB_{\hat\mu'_1\ldots\hat\mu'_r}{}^{\hat\nu'_1\ldots\hat\nu'_s}}{dt}\right], \tag{4.3}$$

where the hatted indices $\hat\mu, (\mu = 2, \ldots D)$ have been introduced, the remaining A-fields have been denoted by B and ξ has been renamed as t.

This theory describes a motion on the coset $\mathcal{G}^{++}/K^{++}_{(+)}$ where $K^{++}_{(+)}$ is the subalgebra of $\mathcal{G}^{++}$ invariant under the Chevalley involution. This 'cosmological' action $S_{\mathcal{G}^{++}_C}$ generalises to all $\mathcal{G}^{++}$ the $E_{10}\,(\equiv E_8^{++})$ action of reference [15, 27] proposed in the context of M-theory and cosmological billiards. Since $K^{++}_{(+)}$ is defined by the Chevalley involution which commutes with the Weyl reflection of $\mathcal{G}^{++}$ this coset admits only the euclidean signature.

4.2 The $\mathcal{G}^{++}_B$-theory

The $\mathcal{G}^{++}_B$-theory is obtained by performing the same truncation as the one of section 4.1, namely we equate as before to zero all the fields in the $\mathcal{G}^{+++}$-invariant action eq. (2.10) which multiply generators involving the root α_1, but this truncation is performed *after* the $\mathcal{G}^{+++}$ Weyl transformation eq. (3.2) which transmutes the time index 1 to a space index. This gives an action $S_{\mathcal{G}^{++}_B}$ which is formally identical to the one given by eqs. (4.1), (4.2) and (4.3) but with a Lorentz signature for the metric, which in the flat coordinates amounts to a negative sign for the Lorentz metric component η_{22}, and with ξ identified to the missing space coordinate instead of t. This theory admits exact solutions identical to intersecting extremal brane solutions of the corresponding maximally oxidised theory smeared in all direction but one [14]. These solutions provide a laboratory to understand the significance of the higher level fields and to check whether or not the Kac-Moody theory can described uncompactified theories [15, 14, 25].

The $S_{\mathcal{G}^{++}_B}$ action is thus characterised by a signature $(1, D-2, +)$ where the sign $+$ means that eq. (3.4) is fulfilled for all the simple positive step operators implying that all the kinetic energy terms in the action are positive. The theory describes a motion on the coset $\mathcal{G}^{++}/K^{++}_{(-)}$. $K^{++}_{(-)}$ is the subalgebra of $\mathcal{G}^{++}$ invariant under the time involution Ω_2 defined as in eq. (2.6) with 2 as the time coordinate and restricted to $\mathcal{G}^{++}$. As this involution will not generically commute with Weyl reflections, the same coset can be described by actions $S^{(T,S,\varepsilon)}_{\mathcal{G}^{++}_{(i_1 i_2 \dots i_t)}}$, where the global signature is (T, S, ε) with ε denoting a set of signs, one for each simple step operator which does not belong to the gravity line, defined by eqs. (3.4) and (3.5), and $i_1 i_2 \dots i_t$ are the time indices. The equivalence of the different actions has been shown by deriving differential equations relating the fields parametrising the different coset representatives [14] and in the special case of $\mathcal{G}^{++}_B = E_8^{++}$ all the signatures in the orbits of $(1, D-2, +)$ has been found and agreed with [16, 17] and [22, 23]. In the next section we derive all the signatures in the orbit of $(1, D-2, \{\epsilon = +\})$ for all $\mathcal{G}^{++}_B$-theories.

5. The signatures of $\mathcal{G}^{++}_B$-theories

We will characterise the different signatures of $\mathcal{G}^{++}_B$ in terms of $\mathcal{G}^{+++}$, namely we will determine all the signatures in the Weyl orbit of $(1, D-1, \{\epsilon = +\})$ with the index 1 fixed to be a space-like coordinate. We first discuss in detail the $\mathcal{G}^{++}_B = A^{++}_{D-3} \subset A^{+++}_{D-3}$ case. The other $\mathcal{G}^{+++}$ contain as a subalgebra A^{+++}_{D-3} (there is always a graviphoton present at some level) consequently the signatures of all $\mathcal{G}^{++}_B$ include at least those of A^{++}_{D-3}.

	sign$(\Omega K^3{}_4)$	sign$(\Omega R^{35\ldots D,D})$	sign$(\Omega K^{D-1}{}_D)$
a.	+	-	+
b.	+	-	-
c.	-	+	+
d.	-	+	-

Table 2: Conditions on Ω's leading to non-trivial signature changes under the Weyl reflection W_{α_D}.

5.1 A_{D-3}^{+++}

5.1.1 $D > 5$

Our purpose is to determine all $S^{(T,S,\varepsilon)}_{A^{++}_{(i_1 i_2 \ldots i_t)}}$ equivalent to $S_{A_B^{++}}$, i.e. all Ω' related to Ω_2 via a Weyl reflection of A_{D-3}^{++} (see eq. (3.1)). As explain above, the only Weyl reflections changing the signature in a non-trivial way are the ones generated by simple roots not belonging to the gravity line. Here there is only one such a simple root , namely α_D (see figure 1). The Weyl reflection W_{α_D} exchanges the following roots,

$$\alpha_{D-1} \leftrightarrow \alpha_{D-1} + \alpha_D \tag{5.1}$$

$$\alpha_3 \leftrightarrow \alpha_3 + \alpha_D\,. \tag{5.2}$$

One can express this Weyl reflection as a conjugaison by a group element $U_{W_{\alpha_D}}$ of A_{D-3}^{+++}. The non-trivial action of $U_{W_{\alpha_D}}$, on the step operators is given by,

$$K^{D-1}{}_D \leftrightarrow \sigma R^{4\ldots D,D-1} \tag{5.3}$$

$$K^3{}_4 \leftrightarrow \rho R^{35\ldots D,D}\,, \tag{5.4}$$

where σ and ρ are +1 or -1 and the tensor $R^{a_4\ldots a_D,a_{D+1}}$ is in the representation[7] $[1,0,\ldots 1,0,0]$ of A_{D-1} that occurs at level one [13]. $R^{4\ldots D,D-1}$ is the generator of A_{D-3}^{+++} associated to the root $\alpha_{D-1}+\alpha_D$ and $R^{35\ldots D,D}$ the one associated to $\alpha_3+\alpha_D$.

In order to obtain a change of signature, we need generically $\Omega K^3{}_4 \neq \Omega' K^3{}_4$ and/or $\Omega K^{D-1}{}_D \neq \Omega' K^{D-1}{}_D$. Using eqs. (5.3), (5.4) and eq. (3.3), these conditions are equivalent to sign$(\Omega K^3{}_4) \neq$ sign$(\Omega R^{35\ldots D,D})$ and/or sign$(\Omega K^{D-1}{}_D) =$ sign$(\Omega R^{4\ldots D,D-1})$. The following equalities,

$$\begin{aligned} \mathrm{sign}(\Omega R^{4\ldots D,D}) &= -\mathrm{sign}(\Omega R^{35\ldots D,D}) \cdot \mathrm{sign}(\Omega K^3{}_4) \\ &= -\mathrm{sign}(\Omega R^{4\ldots D,D-1}) \cdot \mathrm{sign}(\Omega K^{D-1}{}_D)\,, \end{aligned} \tag{5.5}$$

imply that we have only $\Omega K^3{}_4 \neq \Omega' K^3{}_4$ *and* $\Omega K^{D-1}{}_D \neq \Omega' K^{D-1}{}_D$. These inequalities lead to four different possibilities summarised in table 2.

Note that by eq. (5.5) the sign of $\Omega R^{4\ldots D,D-1}$ is deduced from the signs of $\Omega K^3{}_4$, $\Omega R^{35\ldots D,D}$ and $\Omega K^{D-1}{}_D$. All the signatures in the orbit of $(1, D-1, +)$, where + is the sign associated to the generator $R^{a_4\ldots a_D,a_{D+1}}$ defined by eq. (3.4), are now derived.

[7] Here we adopt the following convention : the Dynkin labels of the A_{D-1} representations are labelled from right to left when compared with the labelling of the Dynkin diagram of figure 1. For instance the last label on the right refers to the fundamental weight associated with the root labelled 1 in figure 1. In [13], the opposite convention is used.

	1	2	3	4	...	D-1	D
a.1.	s	t	t	s	...	t	s
a.2.	s	s	s	t	...	t	s
a.3.	s	s	t	s	...	s	t
a.4.	s	t	s	t	...	s	t

Table 3: Ω's leading to signature changes under the Weyl reflection W_{α_D}. Space-like (resp. time-like) coordinate are denoted by s (resp. t).

	signature Ω'	conditions on Ω
a.1.	$(S, T, (-)^D)$	$T \geq 3$ and $S \geq 3$
a.2.	$(S-4, T+4, (-)^D)$	$T \geq 3$ and $S \geq 4$
a.3.	$(S, T, (-)^D)$	$T \geq 3$ and $S \geq 4$
a.4.	$(S, T, (-)^D)$	$T \geq 3$ and $S \geq 3$

Table 4: Ω''s obtained by the Weyl reflection W_{α_D} from Ω's given in table 3.

- *Step 1*: Let us first consider a Ω characterised by a signature $(T, S, +)$ where T is odd. We consider this set of signatures because it contains our starting point $(1, D-1, +)$ and also because other signatures of this type will be useful for the recurrence, i.e. the signature obtained from $(1, D-1, +)$ will lead in some cases to new signatures of the general type $(T, S, +)$ where T is odd. We analyse the different possibilities of table 2.

  - a. The coordinates 3 and 4 are of different nature as well as the coordinates $D-1$ and D. Moreover they are an even number of time coordinates in the set $\{3, 5, \ldots, D-1\}$ as a direct consequence of the sign $-$ in the second column of table 2 and the fact that $R^{a_4 \cdots a_D, a_{D+1}}$ satisfy eq. (3.4) . So they are an odd number of time coordinates in the complementary subset $\{1, 2, 4, D\}$, i.e. 1 or 3. These conditions lead to the possibilities given in table 3.

    The nature of the coordinates 1, 2, 3 and D (resp. 4,...,$D-1$) does not (resp. does) change under the action of the Weyl reflection generated by α_D on Ω's satisfying the conditions a given in table 3. We must also determine which sign eq. (3.4) or eq. (3.5) characterises $R^{a_1 \cdots a_{D-3}, a_{D-2}}$ under the action of the conjugated involution Ω' given by eq. (3.1). Using eq. (3.6) one has $\text{sign}(\Omega' R^{4 \ldots D, D}) = \text{sign}(\Omega R^{4 \ldots D, D}) = -1$ because there is an odd number of time coordinates in $\{4, \ldots, D-1\}$ before Weyl reflection. The nature of all these coordinates changes under the action of W_{α_D}. Therefore if D is even, we have an odd number of time coordinates in this subset and $\text{sign}(\Omega' R^{4 \ldots D, D})$ satisfies eq. (3.4). If D is odd we get an even number of time coordinates yielding to the other sign eq. (3.5). Therefore, the action of W_{α_D} on the Ω's characterised by the signatures of table 3 yields Ω''s characterised by the signatures given in table 4.

  - b. We get the same possibilities as those of case a (see table 3) except for the

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	signature Ω'	conditions on Ω
b.1.	$(S, T, (-)^D)$	$T \geq 3$ and $S \geq 4$
b.2.	$(S-4, T+4, (-)^D)$	$T \geq 1$ and $S \geq 5$
b.3.	$(S, T, (-)^D)$	$T \geq 3$ and $S \geq 3$
b.4.	$(S, T, (-)^D)$	$T \geq 5$ and $S \geq 2$

Table 5: Ω''s obtained by the Weyl reflection W_{α_D} from Ω's given in table 3 with the nature of the coordinate $D-1$ changed.

	1	2	3	4	...	D-1	D
c.1.	s	s	s	s	...	t	s
c.2.	s	t	t	t	...	t	s
c.3.	s	t	s	s	...	s	t
c.4.	s	s	t	t	...	s	t

Table 6: Ω's leading to signature changes under the Weyl reflection W_{α_D} in case c. There is an odd number of time coordinates in the subset $\{4, \ldots, D-1\}$.

	signature Ω'	conditions on Ω
c.1.	$(S-4, T+4, (-)^D)$	$T \geq 1$ and $S \geq 5$
c.2.	$(S, T, (-)^D)$	$T \geq 5$ and $S \geq 2$
c.3.	$(S, T, (-)^D)$	$T \geq 3$ and $S \geq 4$
c.4.	$(S, T, (-)^D)$	$T \geq 3$ and $S \geq 3$

Table 7: Ω''s obtained by the Weyl reflection W_{α_D} from Ω's given in table 6..

coordinate $D-1$ which is different. Therefore the new signatures will be the same. Only the conditions on S and T can differ. These conditions are given in table 5.

– c. The coordinates 3 and 4 are of the same nature, $D-1$ and D are of different nature. Moreover there are an odd number of time coordinates in $\{3, 5, \ldots, D-1\}$ and therefore an even number of time coordinates in the complementary subset $\{1, 2, 4, D\}$, i.e. 0, 2 (1 being always space-like in $\mathcal{G}_B^{++}$). These conditions lead to the possibilities given in table 6. The new signatures are given in table 7 (the sign for $R^{a_1 \ldots a_{D-3}, a_{D-4}}$ is determined by applying a similar reasoning as the one developed in case a).

– d. We get the same signature as those of case c (see table 6) except for the nature of the coordinate $D-1$. Again, only the conditions on T and S can differ. The new signatures are given in table 8.

Summary: from a signature $(T, S, +)$ with T odd, we can reach the signatures

$$\begin{aligned} (S-4, T+4, (-)^D) \quad &\text{if } \{T \geq 3 \text{ and } S \geq 4\} \text{ or } \{T \geq 1 \text{ and } S \geq 5\} \\ (S, T, (-)^D) \quad &\text{if } \{T \geq 5 \text{ and } S \geq 2\} \text{ or } \{T \geq 3 \text{ and } S \geq 3\}. \end{aligned} \tag{5.6}$$

	signature Ω'	conditions on Ω
d.1.	$(S-4, T+4, (-)^D)$	$T \geq 1$ and $S \geq 6$
d.2.	$(S, T, (-)^D)$	$T \geq 3$ and $S \geq 3$
d.3.	$(S, T, (-)^D)$	$T \geq 3$ and $S \geq 3$
d.4.	$(S, T, (-)^D)$	$T \geq 5$ and $S \geq 2$

Table 8: Ω''s obtained by the Weyl reflection W_{α_D} from Ω's given in Table 6 with the nature of the coordinate $D-1$ changed.

In order to find the Weyl orbits of $(1, D-1, +)$ we need to distinguish between D even and D odd.

D **even:** the conditions given by eq. (5.6) simplify to

$$\begin{aligned} (S-4, T+4, +) \quad &\text{if} \;\; \{T \geq 1 \text{ and } S \geq 5\} \\ (S, T, +) \quad &\text{if} \;\; \{T \geq 3 \text{ and } S \geq 3\}\,. \end{aligned} \tag{5.7}$$

If we start from $(1, D-1, +)$, after the action of W_{α_D} one gets $(D-5, 5, +)$ which is of the generic type $(T, S, +)$ with T odd furthermore all the other signatures that we reach given by eq. (5.7) are of this type. Therefore we can use the above analysis and taking into account the conditions eq. (5.7) we conclude that when D is even, the signatures in the W_{α_D} orbit of $(1, D-1, +)$ for $\mathcal{G}_B^{++} = A_{D-3}^{++}$ are given by (n is an integer)

$$\begin{aligned} &(1, D-1, +) \\ &(1+4n, D-1-4n, +) \qquad 3 \leq 4n+1 \leq D-3 \\ &(D-1-4n, 1+4n, +) \qquad 5 \leq 4n+1 \leq D-1\,. \end{aligned} \tag{5.8}$$

D **odd:** after the action of W_{α_D} the signature $(D-5, 5, -)$ is not of the type $(T, S, +)$ with T odd. To determine the orbit of $(1, D-1, +)$, we have thus to analyse the signatures of the form $(T, S, -)$ with T even and S odd.

- *Step 2:* let us consider an involution Ω characterised by a signature $(T, S, -)$ where T is even and S is odd (D is odd). The discussion of the different possible cases of signature change is similar to the ones discussed in step 1. Indeed, the even number of time coordinates balances the minus sign for the kinetic term of $R^{a_1 \cdots a_{D-3}, a_{D-4}}$ (see eq. (3.5)). We must be careful as far as the conditions on T and S are concerned. The only difference with the tables 4, 5, 7, 8 is that we have for D odd the opposite parity for the number of times in the coordinates $\{4, \ldots, D-1\}$. Therefore, starting with $(T, S, -)$ we will reach the signature $(S, T, +)$ under the following conditions,

$$(S, T, +) \;\; \text{if} \;\; \{T \geq 2 \text{ and } S \geq 5\} \text{ or } \{T \geq 4 \text{ and } S \geq 3\}. \tag{5.9}$$

The sign + given by eq. (3.4) for $\Omega' R^{a_1 \cdots a_{D-3}, a_{D-4}}$ is obtained by a reasoning similar to the one given below table 3 taking into account the fact that the number of dimensions is odd.

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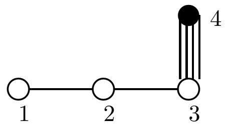

Figure 2: Dynkin diagram of A_1^{+++}.

The new signature eq. (5.9) is of the type considered in step 1. The conditions for getting new signatures by acting again with W_{α_D} on this signature can thus be deduced from eq. (5.6). Starting from $(1, D-1, +)$ the step 1 gives us $(D-5, 5, -)$, then step 2 gives us $(5, D-5, +)$, step 1 can be used again to obtain $(D-9, 9, -)$. Repeating the argument, all the new signatures are obtained using "step 1" or "step 2".

We conclude that when D is odd, the following signatures can be reach,

$$\begin{aligned} &(1+4n, D-1-4n, +) \quad 1 \leq 4n+1 \leq D-2 \\ &(D-1-4n, 1+4n, -) \quad 1 < 4n+1 \leq D\,. \end{aligned} \tag{5.10}$$

We can rewrite eq. (5.8) and eq. (5.10) in a more concise way and conclude that for *all* D (odd and even) the signatures of $\mathcal{G}_B^{++}$ in the Weyl orbit of $(1, D-1, +)$ are given by

$$\begin{aligned} (1+4n, D-1-4n, +) &\quad 0 \leq n \leq \left[\frac{D-3}{4}\right] \\ (D-1-4n, 1+4n, (-)^D) &\quad 1 < n \leq \left[\frac{D-1}{4}\right], \end{aligned} \tag{5.11}$$

where n is an integer and $[x]$ is the integer part of x.

5.1.2 $D = 5$

The reasoning of the previous section cannot apply for $D = 5$ because the conditions on Ω to obtain new signatures Ω' given in table 2 assumed $4 < D-1$. These conditions give in this case $\mathrm{sign}(\Omega K^3{}_4) \neq \mathrm{sign}(\Omega R^{35,5})$ *and* $\mathrm{sign}(\Omega K^4{}_5) \neq \mathrm{sign}(\Omega R^{45,4})$. Starting from a signature $(1, 4, +)$, this implies that the coordinate 4 must be the time. We get by acting with the Weyl reflection generated by α_5 the signature $(0, 5, -)$. Consequently all the possible signatures for $D = 5$ are

$$(1, 4, +)\ (0, 5, -)\,. \tag{5.12}$$

5.1.3 $D = 4$

The Dynkin diagram of A_1^{+++} is depicted in figure 2.

To get a signature change due to the action of the Weyl reflection generated by α_4 we need $\mathrm{sign}(\Omega K^3{}_4) \neq \mathrm{sign}(\Omega R^{3,4})$ which is not possible since we started from $(1, 3, +)$. The symmetric tensor $R^{a,b}$ is the representation [2,0,0] of A_3 that occurs at level one. The only possible signature is the minkowskian one,

$$(1, 3, +)\,. \tag{5.13}$$

	sign($\Omega K^4{}_5$)	sign($\Omega R^{46\ldots D}$)
A	+	-
B	-	+

Table 9: Conditions on Ω's leading to non-trivial signature change under the Weyl reflection W_{α_D} for $S_{D^{++}_{D-2}}$.

5.2 D^{+++}_{D-2}

5.2.1 $D > 6$

They are two simple roots not belonging to the gravity line, namely α_D and α_{D+1} (see figure 1). Given a signature (T, S, ε), the first sign in the set ε is associated to the generator[8] $R^{a_1 a_2}$ and the second one to the generator $R^{a_5 \cdots a_D}$ (see eqs. (3.4) and (3.5)). We will analyse the possible signature changes due to W_{α_D} and $W_{\alpha_{D+1}}$.

1. The Weyl reflection generated by the root α_D will never *non trivially* change the signature. Indeed, its action on the simple root α_{D-2} and the corresponding action of $U_{W_{\alpha_D}}$ on the simple generator $R^{D-2\,D}$ are,

$$W_D\colon \quad \alpha_{D-2} \quad \leftrightarrow \quad \alpha_{D-2} + \alpha_D \quad U_{W_D}\colon \quad K^{D-2}{}_{D-1} \quad \leftrightarrow \quad \sigma R^{D-2D}.$$

   Therefore to obtain a change of signature one needs (see eq. (3.1)): sign($\Omega K^{D-2}{}_{D-1}$) $\neq$ sign(ΩR^{D-2D}). We have

$$\mathrm{sign}(\Omega R^{D-1D}) = -\mathrm{sign}(\Omega R^{D-2D}).\mathrm{sign}(\Omega K^{D-2}{}_{D-1})\,,$$

   thus sign(ΩR^{D-1D}) $= +$ implies that the coordinates $D-1$ and D are of different nature because $R^{D-1\,D}$ satisfies eq. (3.4). So even if $\Omega' K^{D-2}{}_{D-1} \neq \Omega K^{D-2}{}_{D-1}$ we will not have a *non trivial* signature change but just an exchange between the nature of the coordinates $D-1$ and D.

2. We now consider the action of the Weyl reflection generated by the root α_{D+1}. Its action on α_4 is

$$W_{D+1}\colon \quad \alpha_4 \quad \leftrightarrow \quad \alpha_4 + \alpha_{D+1} \quad U_{W_{D+1}}\colon \quad K^4{}_5 \quad \leftrightarrow \quad \rho R^{46\cdots D}.$$

   In order to have a signature change, we need sign($\Omega K^4{}_5$) $\neq$ sign($\Omega R^{46\ldots D}$). This leads to two possibilities explicitly given in table 9. The condition A means that the nature of the coordinates 4 and 5 are different *and* that if we start from $(T, S, \pm, +)$ (resp. $(T, S, \pm, -)$) there is an even (resp. odd) number of times in $\{4, 6, \ldots, D\}$. Whereas condition B means that that the nature of the coordinates 4 and 5 are the same *and* that if we start from $(T, S, \pm, +)$ (resp. $(T, S, \pm, -)$) there is an odd (resp. even) number of times in $\{4, 6, \ldots, D\}$.

[8] The tensor $R^{a_1 a_2}$ is in the representation $[0, 1, 0, \ldots, 0, 0]$ of A_{D-1} that occurs at level $(1, 0)$, the tensor $R^{a_5 \cdots a_D}$ is in the representation $[0, \ldots, 0, 1, 0, 0, 0]$ of A_{D-1} that occurs at level $(0, 1)$ [13]. The level (l_1, l_2) of a root α of D^{+++}_{D-2} can be read in its decomposition in terms of the simple roots $\alpha = m_1\alpha_1 + \cdots + m_{D-1}\alpha_{D-1} + l_1\alpha_D + l_2\alpha_{D+1}$.

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	1	2	3	4	5	Ω'	$T \geq$	$S \geq$
i.	s	s	t	t	s	$(S, T, +, (-)^D)$	3	3
ii.	s	s	s	s	t	$(S-4, T+4, +, (-)^D)$	1	4
iii.	s	t	t	s	t	$(S, T, +, (-)^D)$	3	2

Table 10: Ω's leading to signature changes under the Weyl reflection $W_{\alpha_{D+1}}$ and the related new signatures Ω''s (in the case A).

	1	2	3	4	5	Ω'	$T \geq$	$S \geq$
i.	s	s	s	s	s	$(S-4, T+4, +, (-)^D)$	1	5
ii.	s	s	t	t	t	$(S, T, +, (-)^D)$	3	2
iii.	s	t	t	s	s	$(S, T, +, (-)^D)$	3	3

Table 11: Ω's leading to signature changes under the Weyl reflection $W_{\alpha_{D+1}}$ and the related new signatures Ω''s (in the case **B**).

- By analogy with the A_{D-3}^{+++} case, we start from the signature $(T, S, +, +)$ where T is an odd number.
  A. There is an odd number of times in $\{1, 2, 3, 5\}$, i.e. 1 or 3. These time coordinates can be distributed as in table 10. The new signatures Ω', also given in Table 10, are deduced from the fact that the action of the Weyl reflection generated by α_{D+1} will change the nature of the coordinates greater than 4.
  B. There is an even number of times in $\{1, 2, 3, 5\}$ i.e. 0, 2 or 4. 4 is excluded here because we want that 1 is space-like. These time coordinates can be distributed as in table 11. The new signatures are also given in table 11.
  $\Rightarrow$ if D is even, from a signature $(T, S, +, +)$ we can reach the signatures $(S, T, +, +)$ (if $T \geq 3$ and $S \geq 3$) and $(S-4, T+4, +, +)$ (if $T \geq 1$ and $S \geq 5$).
  $\Rightarrow$ if D is odd, from a signature $(T, S, +, +)$ we can reach the signatures $(S, T, +, -)$ (if $T \geq 3$ and $S \geq 2$) and $(S-4, T+4, +, -)$ (if $T \geq 1$ and $S \geq 4$).
- From the signature $(T, S, +, -)$ where T is an even number and S odd (D is odd). By the same procedure, we can conclude that the signatures $(S, T, +, +)$ can be reached (if $T \geq 2$ and $S \geq 3$) and the signatures $(S-4, T+4, +, +)$ (if $S \geq 5$).

Summary: we get the same signatures as the ones of pure gravity as expected since W_{α_D} does not change the signature and only $W_{\alpha_{D+1}}$ has a non-trivial action,

$$\begin{aligned} &(1+4n, D-1-4n, +, +) \quad 0 \leq n \leq \left[\frac{D-3}{4}\right] \\ &(D-1-4n, 1+4n, +, (-)^D) \quad 1 < n \leq \left[\frac{D-1}{4}\right], \end{aligned} \tag{5.14}$$

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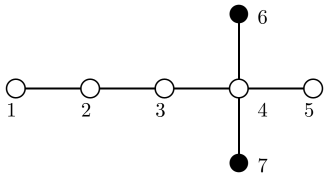

Figure 3: Dynkin diagram of D_4^{+++}.

where $[x]$ is the integer part of x. We could have acted with the Weyl reflection generated by the graviphoton lying at level $(1,1)$ instead of acting with $W_{\alpha_{D+1}}$ and we would have obtained the same results. The sign of the kinetic term of the graviphoton agrees with eq. (5.11).

5.2.2 $D = 6$

They are two simple roots out of the gravity line, namely α_6 and α_7 (see figure 3).

Given a signature (T, S, ε), the first sign in the set ε is associated to the generator $R^{a_1 a_2}$ and the second one to the generator $\tilde{R}^{a_1 a_2}$.

We can immediately conclude that no signature changes are possible. Indeed the Weyl reflections generated by the two simple roots not belonging to the gravity line cannot change the signature in the same way as the Weyl reflection W_{α_D} cannot do it in the previous section.

5.3 E_6^{+++}

They are two simple roots not belonging to the gravity line, α_8 and α_9 (see figure 1). The non trivial actions of the corresponding Weyl reflections W_8, W_9 are

$$\begin{array}{llcllrcl} W_8: & \alpha_5 & \leftrightarrow & \alpha_5+\alpha_8 & U_{W_8}: & K^5{}_6 & \leftrightarrow & \rho R^{578} \\ & \alpha_9 & \leftrightarrow & \alpha_9+\alpha_8 & & R & \leftrightarrow & \sigma \tilde{R}^{678} \\ W_9: & \alpha_8 & \leftrightarrow & \alpha_8+\alpha_9 & U_{W_9}: & R^{678} & \leftrightarrow & \delta \tilde{R}^{678}. \end{array}$$

The tensor R^{abc} is the representation $[0,0,1,0,0,0,0]$ of A_7 that occurs at level[9] $(1,0)$, the tensor R is the representation [0,0,0,0,0,0,0] of A_7 that occurs at level $(0,1)$ and $\tilde{R}^{abc}$ is the representation $[0,0,1,0,0,0,0]$ of A_7 that occurs at level $(1,1)$ [13]. The sign of the kinetic term of $\tilde{R}^{abc}$ can be deduced from the ones of R^{abc} and R, it is the product of these two signs.

The Weyl reflection generated by α_9 exchanges the signs of R^{abc} and $\tilde{R}^{abc}$.

To obtain a signature change from $(1,7,+,+)$,[10] we must act with the Weyl reflection generated by α_8 and there must be an odd number of time coordinates in the following subset: 6,7,8. In all of this cases, the new signature is $(2,6,-,-)$. Now we can start from this new signature and act with the Weyl reflection generated by α_9 to get the signature

[9]The level (l_1, l_2) of a root α of E_6^{+++} can be read in its decomposition in terms of the simple roots $\alpha = m_1\alpha_1 + \cdots + m_7\alpha_7 + l_1\alpha_8 + l_2\alpha_9$.

[10]The first sign characterises R^{abc} and the second one characterises R (see eqs. (3.4) and (3.5)).

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$(2,6,+,-)$ or with the one generated by α_8 to get $(5,3,+,+)$. Acting again with the Weyl reflections generated by α_8 (vertical arrows) and α_9 (horizontal arrows) on these signatures, we can conclude that all the signatures are

$$\begin{array}{ccc}
(1,7,+,+) & & \\
\downarrow & & \\
(2,6,-,-) & \rightarrow & (2,6,+,-) \\
\downarrow & & \downarrow \\
(5,3,+,+) & & (3,5,-,+) \\
\downarrow & & \downarrow \\
(6,2,-,-) & \rightarrow & (6,2,+,-).
\end{array}$$

Theses signatures are the expected ones from the gravity, i.e. $(1,7,+,+)$, $(3,5,-,+)$ and $(5,3,+,+)$, plus new ones.

5.4 E_7^{+++}

They are two simple roots not belonging to the gravity line, α_9 and α_{10}. The non trivial actions of the Weyl reflections W_9, W_{10} are

$$\begin{array}{rcclrccl}
W_9\colon & \alpha_8 & \leftrightarrow & \alpha_8+\alpha_9 & U_{W_9}\colon & K^8{}_9 & \leftrightarrow & \rho R^8 \\
W_{10}\colon & \alpha_6 & \leftrightarrow & \alpha_6+\alpha_{10} & U_{W_{10}}\colon & K^6{}_7 & \leftrightarrow & \delta R^{689}.
\end{array}$$

The tensor R^a is the representation $[1,0,0,0,0,0,0,0]$ of A_8 that occurs at level[11] $(1,0)$, the tensor R^{abc} is the representation [0,0,1,0,0,0,0,0] of A_8 that occurs at level $(0,1)$.

The Weyl reflection generated by the root α_9 will change the signature if $\mathrm{sign}(\Omega K^8{}_9) \neq \mathrm{sign}(\Omega R^8)$. The Weyl reflection generated by the root α_{10} will change the signature if $\mathrm{sign}(\Omega K^6{}_7) \neq \mathrm{sign}(\Omega R^{689})$. With these rules we get the following signatures (a horizontal arrow represents the action of the Weyl reflection generated by α_{10} and a vertical one the action of the Weyl reflection generated by α_9),

$$\begin{array}{ccccccc}
(1,8,+,+) & \rightarrow & (2,7,-,-) & \rightarrow & (5,4,+,+) & \rightarrow & (6,3,-,-) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(0,9,-,-) & \rightarrow & (3,6,+,+) & \rightarrow & (4,5,-,-) & \rightarrow & (7,2,+,+).
\end{array}$$

The signs in the above signatures refer to the kinetic terms of R^a and R^{abc} (see eqs. (3.4) and (3.5)).

5.5 E_8^{+++}

The signatures are given in [14]. To be complete, we recall them here,

$$(1,10,+) \rightarrow (2,9,-) \rightarrow (5,6,+) \rightarrow (6,5,-) \rightarrow (9,2,+)\,. \tag{5.15}$$

The sign refers to the sign of the kinetic term of R^{abc} which is the representation [0,0,1,0,0,0,0,0,0,0] of A_{10} that occurs at level 1.

[11]The level (l_1,l_2) of a root α of E_7^{+++} can be read in its decomposition in terms of the simple roots $\alpha = m_1\alpha_1 + \cdots + m_7\alpha_7 + m_8\alpha_8 + l_1\alpha_9 + l_2\alpha_{10}$.

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6. Non simply laced algebras

6.1 B_{D-2}^{+++}

The only non trivial action of the Weyl reflection generated by the short root α_D is

$$W_D\colon\ \alpha_{D-1} \quad\leftrightarrow\quad \alpha_{D-1}+2\alpha_D \quad U_{W_D}\colon\ K^{D-1}{}_D \quad\leftrightarrow\quad \sigma R^{D-1D}.$$

The tensor R^{ab} is the representation $[0,1,0,,\dots,0]$ of A_{D-1} that occurs at level[12] $(2,0)$ [13]. To get a signature change with this Weyl reflection we need $\mathrm{sign}(\Omega K^{D-1}{}_D) \neq \mathrm{sign}(\Omega R^{D-1D})$ which is impossible since the sign of the kinetic term of R^{ab} is given by eq. (3.4).

We are therefore left with one simple long root, namely α_{D+1}. The Weyl reflection $W_{\alpha_{D+1}}$ will clearly produce the same signature changes as $W_{\alpha_{D+1}}$ does for D_{D-2}^{+++}. Therefore all possible signatures are the ones found for D_{D-2}^{+++}, i.e. the signatures of pure gravity.

$$\begin{aligned} (1+4n, D-1-4n, +, +) \quad & 0 \leq n \leq \left[\frac{D-3}{4}\right] \\ (D-1-4n, 1+4n, +, (-)^D) \quad & 1 < n \leq \left[\frac{D-1}{4}\right], \end{aligned} \tag{6.1}$$

where $[x]$ is the integer part of x. The first sign refers to the kinetic term of R^a and the second to the one of $R^{a_5\cdots a_D}$. R^a is the representation [1,0,...,0] of A_{D-1} that occurs at level $(1,0)$ and $R^{a_5\cdots a_D}$ the representation [0,...,0,1,0,0,0] that occurs at level (0,1) [13].

6.2 C_{q+1}^{+++}

Clearly, if the Weyl reflections associated to α_4 and α_5 do not change the signature, there will be no signature changes. Let us first have a look at α_5,

$$\begin{array}{lcccllcl} W_5\colon & \alpha_4 & \leftrightarrow & \alpha_4+\alpha_5 & U_{W_5}\colon & R^4 & \leftrightarrow & \rho\tilde{R}^4 \\ & \alpha_6 & \leftrightarrow & \alpha_6+\alpha_5 & & \tilde{R} & \leftrightarrow & \sigma\tilde{\tilde{R}}. \end{array}$$

The tensor R^a is the representation $[1,0,0]$ of A_3 that occurs at level[13] $(1,0,\dots,0)$, the tensor $\tilde{R}^a$ is the representation [1,0,0] of A_3 that occurs at level $(1,1,0,\dots,0)$, the tensor $\tilde{R}$ is the representation $[0,0,0]$ of A_3 that occurs at level (0,0,1,0,...,0) and $\tilde{\tilde{R}}$ is the representation $[0,0,0]$ of A_3 that occurs at level $(0,1,1,0,\dots,0)$. Because we started with a signature where all the generators have the sign of their kinetic term positive, we will not reach new signatures (i.e., here meaning new signs for the kinetic term of R^a or $\tilde{R}$) with this reflection.

Let us now look at α_4,

$$\begin{array}{lcccllcl} W_4\colon & \alpha_5 & \leftrightarrow & \alpha_4+\alpha_5 & U_{W_4}\colon & R & \leftrightarrow & \tilde{R}^4 \\ & \alpha_3 & \leftrightarrow & \alpha_3+2\alpha_4 & & K^3{}_4 & \leftrightarrow & R^{34}. \end{array}$$

[12]The level (l_1, l_2) of a root α of B_{D-2}^{+++} can be read in its decomposition in terms of the simple roots $\alpha = m_1\alpha_1 + \cdots + m_{D-1}\alpha_{D-1} + l_1\alpha_D + l_2\alpha_{D+1}$.

[13]The level $(l_{q+1}, l_q, \dots, l_1)$ of a root α of C_{q-1}^{+++} can be read in its decomposition in terms of the simple roots $\alpha = m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 + l_{q+1}\alpha_4 + \cdots + l_1\alpha_{q+4}$.

(l_1, l_2)	A_5 weight	Tensor
(0,1)	[0,0,0,0,0]	R
(1,0)	[1,0,0,0,0]	R^a
(1,1)	[1,0,0,0,0]	$\tilde{R}^a$
(2,0)	[0,1,0,0,0]	R^{ab}

Table 12: Level decomposition of F_4^{+++}.

The tensor R is the representation [0,0,0] of A_3 that occurs at level (0,1,0,. . . ,0), R^{ab} is the representation [0,1,0] of A_3 that occurs at level (2,0,. . . ,0). To get a signature change we need $\Omega(K^3{}_4) \neq \Omega(R^{34})$ which is impossible because we have eq. (3.4) for R^{34}.

6.3 F_4^{+++}

There are two short simple roots not belonging to the gravity line, namely α_6 and α_7. The generators associated to these simple roots are given in table 12 respectively at level[14] (1,0) and (0,1) [13].

The non trivial action of the Weyl reflection W_{α_6} (and of the corresponding conjugaison by a group element $U_{W_{\alpha_6}}$) on the simple roots (and on the corresponding simple generators) are,

$$\begin{array}{lcclcccl} W_6\colon & \alpha_5 & \leftrightarrow & \alpha_5 + 2\alpha_6 & U_{W_6}\colon & K^5{}_6 & \leftrightarrow & R^{56} \\ & \alpha_7 & \leftrightarrow & \alpha_7 + \alpha_6 & & R & \leftrightarrow & \tilde{R}^6. \end{array}$$

W_{α_6} can not change the signature because it needs $\mathrm{sign}(\Omega K^5{}_6) \neq \mathrm{sign}(\Omega R^{56})$ which is impossible since the sign of the kinetic term for R^{ab} is always positive. Moreover W_{α_6} can neither change the sign of the kinetic term of R because we start from a signature such that all kinetic terms are characterised by eq. (3.4). The action of W_{α_7} on the simple roots is,

$$W_7\colon \quad \alpha_6 \quad \leftrightarrow \quad \alpha_6 + \alpha_7 \quad U_{W_7}\colon \quad R^6 \quad \leftrightarrow \quad \tilde{R}^6.$$

This reflection can only *a priori* change the sign of the kinetic term for R^a. This sign can change if $\mathrm{sign}(\Omega R^6) \neq \mathrm{sign}(\Omega \tilde{R}^6)$ which is impossible since we started from a signature with all the signs given by eq. (3.4). Therefore the only possible signature is

$$(1, 5, +, +), \tag{6.2}$$

where the first sign refers to R^a and the second to R.

6.4 G_2^{+++}

There is only one simple root not belonging to the gravity line, namely α_5. The only non trivial action of the Weyl reflection generated by α_5 on the simple roots is (see table 13)

$$W_5\colon \alpha_4 \quad \leftrightarrow \quad \alpha_4 + 3\alpha_5 \quad U_{W_5}\colon \quad K^4{}_5 \quad \leftrightarrow \quad \tilde{R}^{455}.$$

[14]The level (l_1, l_2) of a root α of F_4^{+++} can be read in its decomposition in terms of the simple roots $\alpha = m_1\alpha_1 + \cdots + m_5\alpha_5 + l_1\alpha_6 + l_2\alpha_7$.

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l	A_4 weight	Tensor
1	[1,0,0,0]	R^a
3	[1,1,0,0]	$\tilde{R}^{abc}$

Table 13: Level decomposition of G_2^{+++}.

To get a signature change we need $\text{sign}(\Omega K^4{}_5) \neq \text{sign}(\Omega(\tilde{R}^{455})$, i.e. 5 is a time coordinate (because we start with the sign of $\tilde{R}^{abc}$ given by eq. (3.4)). The new signature is the euclidian one $(0, 5, -)$. These are the only signatures we can reach in agreement with pure gravity in $D = 5$,

$$(1, 4, +) \qquad (0, 5, -)\,, \tag{6.3}$$

where the sign refers to the kinetic term of R^a.

Acknowledgments

One of us (L.H.) would like to warmly thank François Englert for many enjoyable and fruitful conversations. We are grateful to Marc Henneaux and Arjan Keurentjes for interesting comments. This work was supported in part by the NATO grant PST.CLG.979008, by the "Interuniversity Attraction Poles Programme – Belgian Science Policy ", by IISN-Belgium (convention 4.4505.86) and by the European Commission FP6 programme MRTN-CT-2004-005104, in which S.D., L.H. and N.T. are associated to the V.U.Brussel (Belgium).

References

[1] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Higher-dimensional origin of $D = 3$ coset symmetries*, hep-th/9909099.

[2] M.R. Gaberdiel, D.I. Olive and P.C. West, *A class of lorentzian Kac-Moody algebras*, *Nucl. Phys.* **B 645** (2002) 403 [hep-th/0205068].

[3] B. Julia, *Kac-Moody symmetry of gravitation and supergravity theories*, Proc. AMS-SIAM Chicago meeting July 1982, Lecture in Applied Mathematics **21** (1985) 355; *Group disintegrations*, in *Superspace and supergravity*, S.W. Hawking and M. Rocek eds., Cambridge University Press, 1981.

[4] P.C. West, *E_{11} and M-theory*, *Class. and Quant. Grav.* **18** (2001) 4443 [hep-th/0104081].

[5] N.D. Lambert and P.C. West, *Coset symmetries in dimensionally reduced bosonic string theory*, *Nucl. Phys.* **B 615** (2001) 117 [hep-th/0107209].

[6] F. Englert, L. Houart, A. Taormina and P. West, *The symmetry of M-theories*, *JHEP* **09** (2003) 020 [hep-th/0304206].

[7] T. Damour, M. Henneaux and H. Nicolai, *Cosmological billiards*, *Class. and Quant. Grav.* **20** (2003) R145 [hep-th/0212256].

[8] T. Damour and M. Henneaux, *E_{10}, BE_{10} and arithmetical chaos in superstring cosmology*, *Phys. Rev. Lett.* **86** (2001) 4749 [hep-th/0012172];
T. Damour, M. Henneaux, B. Julia and H. Nicolai, *Hyperbolic Kac-Moody algebras and chaos in Kaluza-Klein models*, *Phys. Lett.* **B 509** (2001) 323 [hep-th/0103094].

[9] T. Damour, S. de Buyl, M. Henneaux and C. Schomblond, *Einstein billiards and overextensions of finite-dimensional simple Lie algebras*, *JHEP* **08** (2002) 030 [hep-th/0206125].

[10] F. Englert and L. Houart, *G^{+++} invariant formulation of gravity and M-theories: exact BPS solutions*, *JHEP* **01** (2004) 002 [hep-th/0311255].

[11] P. West, *Very extended E_8 and A_8 at low levels, gravity and supergravity*, *Class. and Quant. Grav.* **20** (2003) 2393 [hep-th/0212291].

[12] H. Nicolai and T. Fischbacher, *Low level representations for E_{10} and E_{11}*, hep-th/0301017.

[13] A. Kleinschmidt, I. Schnakenburg and P. West, *Very-extended Kac-Moody algebras and their interpretation at low levels*, *Class. and Quant. Grav.* **21** (2004) 2493 [hep-th/0309198].

[14] F. Englert, M. Henneaux and L. Houart, *From very-extended to overextended gravity and M-theories*, *JHEP* **02** (2005) 070 [hep-th/0412184].

[15] T. Damour, M. Henneaux and H. Nicolai, *E_{10} and a 'small tension expansion' of M-theory*, *Phys. Rev. Lett.* **89** (2002) 221601 [hep-th/0207267].

[16] A. Keurentjes, *E_{11}: sign of the times*, *Nucl. Phys.* **B 697** (2004) 302 [hep-th/0402090].

[17] A. Keurentjes, *Time-like T-duality algebra*, *JHEP* **11** (2004) 034 [hep-th/0404174].

[18] F. Englert and L. Houart, *G^{+++} invariant formulation of gravity and M-theories: exact intersecting brane solutions*, *JHEP* **05** (2004) 059 [hep-th/0405082].

[19] R. Argurio, F. Englert and L. Houart, *Intersection rules for p-branes*, *Phys. Lett.* **B 398** (1997) 61 [hep-th/9701042].

[20] S. Elitzur, A. Giveon, D. Kutasov and E. Rabinovici, *Algebraic aspects of matrix theory on T^d*, *Nucl. Phys.* **B 509** (1998) 122 [hep-th/9707217];
N.A. Obers and B. Pioline, *U-duality and M-theory, an algebraic approach*, hep-th/9812139.

[21] T. Banks, W. Fischler and L. Motl, *Dualities versus singularities*, *JHEP* **01** (1999) 019 [hep-th/9811194].

[22] C.M. Hull, *Timelike T-duality, de Sitter space, large-N gauge theories and topological field theory*, *JHEP* **07** (1998) 021 [hep-th/9806146];
C.M. Hull and R.R. Khuri, *Branes, times and dualities*, *Nucl. Phys.* **B 536** (1998) 219 [hep-th/9808069].

[23] C.M. Hull, *Duality and the signature of space-time*, *JHEP* **11** (1998) 017 [hep-th/9807127].

[24] B. Julia, *Kac-Moody symmetry of gravitation and supergravity theories*, *Lectures in applied mathematics*, vol. **21** (1985) 335;
A. Sen, *Strong - weak coupling duality in four-dimensional string theory*, *Int. J. Mod. Phys.* **A 9** (1994) 3707 [hep-th/9402002]; *Strong-weak coupling duality in three-dimensional string theory*, *Nucl. Phys.* **B 434** (1995) 179 [hep-th/9408083].

[25] T. Damour and H. Nicolai, *Higher order M-theory corrections and the kac-moody algebra E_{10}*, hep-th/0504153.

[26] A. Kleinschmidt and P. West, *Representations of G^{+++} and the role of space-time*, *JHEP* **02** (2004) 033 [hep-th/0312247].

[27] T. Damour and H. Nicolai, *Eleven dimensional supergravity and the E_{10}/KE_{10} sigma-model at low A_9 levels*, hep-th/0410245.